Please check the examination details below before entering your candidate information

Candidate surname	Other names

Centre Number | Candidate Number

Pearson Edexcel Level 3 GCE

Time 1 hour 30 minutes

Paper reference **9FM0/3A**

Further Mathematics
Advanced
PAPER 3A: Further Pure Mathematics 1

You must have:
Mathematical Formulae and Statistical Tables (Green), calculator

Total Marks

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for symbolic algebraic manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 9 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

Turn over

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Q:1/1/1/1/

P 6 5 4 9 7 A 0 1 3 2

Pearson

1. An ellipse has equation $\frac{x^2}{16} + \frac{y^2}{4} = 1$ and eccentricity e_1

A hyperbola has equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and eccentricity e_2

Given that $e_1 \times e_2 = 1$

(a) show that $a^2 = 3b^2$

(4)

Given also that the coordinates of the foci of the ellipse are the same as the coordinates of the foci of the hyperbola,

(b) determine the equation of the hyperbola.

(3)

a. Firstly calculate e_1 and e_2.

rewriting ellipse eqⁿ : $\frac{x^2}{(4)^2} + \frac{y^2}{(2)^2} = 1$ $\quad a=4,\ b=2$

e_1: $(2)^2 = (4)^2(1-e_1^2)$

$4 = 16(1-e_1^2)$

$\frac{4}{16} = 1 - e_1^2$

$e_1^2 = \frac{12}{16}$

$e_1 = \pm\frac{\sqrt{3}}{2}$

$e_1 \neq -\frac{\sqrt{3}}{2}$

$\therefore e_1 = \frac{\sqrt{3}}{2}$

Conics

	Ellipse	~~Parabola~~	Hyperbola	~~Rectangular Hyperbola~~
Standard Form	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$y^2 = 4ax$	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$xy = c^2$
Parametric Form	$(a\cos\theta, b\sin\theta)$	$(at^2, 2at)$	$(a\sec\theta, b\tan\theta)$ $(\pm a\cosh\theta, b\sinh\theta)$	$\left(ct, \frac{c}{t}\right)$
Eccentricity	$e < 1$ $b^2 = a^2(1-e^2)$	$e = 1$	$e > 1$ $b^2 = a^2(e^2-1)$	$e = \sqrt{2}$
Foci	$(\pm ae, 0)$	$(a, 0)$	$(\pm ae, 0)$	$(\pm\sqrt{2}c, \pm\sqrt{2}c)$
Directrices	$x = \pm\frac{a}{e}$	$x = -a$	$x = \pm\frac{a}{e}$	$x + y = \pm\sqrt{2}c$
Asymptotes	none	none	$\frac{x}{a} = \pm\frac{y}{b}$	$x = 0, y = 0$

$e_1 \times e_2 = 1$

$e_2 = \frac{1}{e_1}$

$e_2 = \frac{1}{\left(\frac{\sqrt{3}}{2}\right)} = \frac{2}{\sqrt{3}}$

Hyperbola eccentricity eqⁿ : $b^2 = a^2(e_2^2 - 1)$

$b^2 = a^2\left(\left(\frac{2}{\sqrt{3}}\right)^2 - 1\right)$

$b^2 = a^2\left(\frac{4}{3} - 1\right)$

$b^2 = a^2(1/3)$ $\quad$ multiply both sides by 3.

$\boxed{3b^2 = a^2}$ // (shown)

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Question 1 continued

b. foci of ellipse = foci of hyperbola

$(\pm ae, 0) \qquad (\pm ae, 0)$

$4 \quad \frac{\sqrt{3}}{2} \qquad$ unknown $\quad \frac{2}{\sqrt{3}}$

$4\left(\frac{\sqrt{3}}{2}\right) = a\left(\frac{2}{\sqrt{3}}\right)$ equate both x-coords and solve for a.

$2\sqrt{3} = \frac{2a}{\sqrt{3}}$ multiply both sides by $\sqrt{3}$!

$6 = 2a$

$a = 3$

$a^2 = 3b^2$ ← shown in part a (sub back in to find b)

$(3)^2 = 3b^2$

$9 = 3b^2$

$3 = b^2$

$\frac{x^2}{9} - \frac{y^2}{3} = 1$

(Total for Question 1 is 7 marks)

P 6 5 4 9 7 A 0 3 3 2

Turn over

2. During 2029, the number of hours of daylight per day in London, H, is modelled by the equation

$$H = 0.3\sin\left(\frac{x}{60}\right) - 4\cos\left(\frac{x}{60}\right) + 11.5 \qquad 0 \leqslant x < 365$$

where x is the number of days after 1st January 2029 and the angle is in radians.

(a) Show that, according to the model, the number of hours of daylight in London on the 31st January 2029 will be 8.13 to 3 significant figures.

(1)

(b) Use the substitution $t = \tan\left(\frac{x}{120}\right)$ to show that H can be written as

$$H = \frac{at^2 + bt + c}{1 + t^2}$$

where a, b and c are constants to be determined.

(2)

(c) Hence determine, according to the model, the date of the first day of 2029 when there will be at least 12 hours of daylight in London.

(4)

a. from 01/01/29 → 31/01/29 30 days

sub in $x = 30$

$H = 0.3\sin\left(\frac{30}{60}\right) - 4\cos\left(\frac{30}{60}\right) + 11.5$

$H \approx 8.13$ (3s.f.) // (shown)

b. given $t = \tan\left(\frac{x}{120}\right)$

use $t = \tan\left(\frac{\theta}{2}\right)$ where $\theta = \frac{x}{60}$ ($t = \tan\frac{\theta}{2}$ more familiar.)

if $t = \tan\frac{\theta}{2}$, $\sin\theta = \frac{2t}{1+t^2}$

$\cos\theta = \frac{1-t^2}{1+t^2}$

proof @ end of question

Question 2 continued

$\sin\left(\frac{x}{60}\right) = \frac{2t}{1+t^2}$

$\cos\left(\frac{x}{60}\right) = \frac{1-t^2}{1+t^2}$

$H = 0.3\left(\frac{2t}{1+t^2}\right) - 4\left(\frac{1-t^2}{1+t^2}\right) + 11.5$

$H = \frac{0.6t}{1+t^2} - \frac{4-4t^2}{1+t^2} + 11.5$

put everything into a similar form

$H = \frac{0.6t - (4-4t^2) + 11.5(1+t^2)}{1+t^2}$

$H = \frac{0.6t - 4 + 4t^2 + 11.5 + 11.5t^2}{1+t^2}$

$H = \frac{15.5t^2 + 0.6t + 7.5}{1+t^2}$ //

$a = 15.5$

$b = 0.6$

$c = 7.5$

c. Sub in $H = 12$ and solve for t.

$12 = \frac{15.5t^2 + 0.6t + 7.5}{1+t^2}$

$12(1+t^2) = 15.5t^2 + 0.6t + 7.5$

$12 + 12t^2 = 15.5t^2 + 0.6t + 7.5$

$3.5t^2 + 0.6t - 4.5 = 0$

solve using quadratic formulae

$t = \frac{-(0.6) \pm \sqrt{(0.6)^2 - (4)(3.5)(-4.5)}}{2(3.5)}$

(Total for Question 2 is 7 marks)

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Turn over

$t = 1.051414214$ or $t = -1.222842785$ (don't round till end)

remember $t = \tan\left(\frac{x}{120}\right)$

$x = 120 \arctan(t)$

$120 \arctan(1.051\ldots) = 97.25468787$

$120 \arctan(-1.22\ldots) = -106$ ← X reject as cannot have -ve. x value since x represents no. of days.

round 97.3 days up so 98 days.

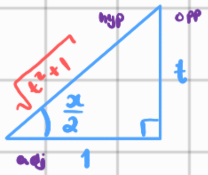
calculate 98 days from Jan.

2029 NOT a leap year

Jan		Feb		March		April		
31	+	28	+	30	+	9	=	98

April 9^{th} 2029

Deriving the t-formulae :



1) draw a right-angled triangle and label angle and sides which you know.

* you are allowed to memorise the t-formulae for sin(x), cos(x), tan(x) and do not have to derive it in the exam unless specifically asked.

2) work out hypotenuse in terms of t, (pythagoras)

$$\sqrt{(t)^2+(1)^2}$$
$$= \sqrt{t^2+1}$$

3) label each side of triangle opposite, adjacent, hypotenuse

4) write out $\sin\left(\frac{x}{2}\right)$, $\cos\left(\frac{x}{2}\right)$, $\tan\left(\frac{x}{2}\right)$ in terms of t.

$$\sin\left(\frac{x}{2}\right) = \frac{t}{\sqrt{t^2+1}} \qquad \sin = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos\left(\frac{x}{2}\right) = \frac{1}{\sqrt{t^2+1}} \qquad \cos = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan\left(\frac{x}{2}\right) = \frac{t}{1} = t \qquad \tan = \frac{\text{opposite}}{\text{adjacent}} \Big/ \frac{\sin}{\cos}$$

5) Now use double-angle formulae and write in terms of t:

$$\sin(x) = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)$$
$$\cos(x) = \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right)$$

$$\sin(x) = 2\left(\frac{t}{\sqrt{t^2+1}}\right)\left(\frac{1}{\sqrt{t^2+1}}\right) \qquad \cos(x) = \left(\frac{1}{\sqrt{t^2+1}}\right)^2 - \left(\frac{t}{\sqrt{t^2+1}}\right)^2$$

$$\sin(x) = \frac{2t}{t^2+1} \qquad \cos(x) = \frac{1-t^2}{1+t^2}$$

$$\tan(x) = \frac{\sin(x)}{\cos(x)} = \frac{\frac{2t}{\cancel{1+t^2}}}{\frac{1-t^2}{\cancel{1+t^2}}} = \frac{2t}{1-t^2}$$

$$\tan(x) = \frac{2t}{1-t^2}$$

3. With respect to a fixed origin O, the points A and B have coordinates $(2, 2, -1)$ and $(4, 2p, 1)$ respectively, where p is a constant.

$\overrightarrow{OA}$

$\overrightarrow{OB}$

For each of the following, determine the possible values of p for which,

(a) OB makes an angle of 45° with the positive x-axis **(3)**

suggests more than 1 p value for each part.

(b) $\overrightarrow{OA} \times \overrightarrow{OB}$ is parallel to $\begin{pmatrix} 4 \\ -p \\ 2 \end{pmatrix}$ **(3)**

(c) the area of triangle OAB is $3\sqrt{2}$ **(3)**

a. $\cos(\theta_x) = \dfrac{x}{|b|}$ only represents x value. $|b|$ represents magnitude of x, y, z coord.

$\cos(45) = \dfrac{4}{\sqrt{(4)^2+(2p)^2+(1)^2}}$

$\dfrac{\sqrt{2}}{2} = \dfrac{4}{\sqrt{4p^2+17}}$

square both sides to remove surd

$\left(\dfrac{\sqrt{2}}{2}\right)^2 = \left(\dfrac{4}{\sqrt{4p^2+17}}\right)^2$

$\dfrac{2}{4} = \dfrac{16}{4p^2+17}$ cross multiply

$2(4p^2+17) = 16(4)$

$8p^2 + 34 = 64$

$8p^2 = 30$

$p^2 = \dfrac{30}{8}$

$p = \pm\sqrt{\dfrac{30}{8}}$ $\qquad \sqrt{\dfrac{30}{8}} = \sqrt{\dfrac{15}{4}} = \dfrac{\sqrt{15}}{\sqrt{4}} = \dfrac{\sqrt{15}}{2}$

$\sqrt{\dfrac{a}{b}} = \dfrac{\sqrt{a}}{\sqrt{b}}$

$p = \pm\dfrac{\sqrt{15}}{2}$ //

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Question 3 continued

parallel means not necessarily $(4, -p, 2)$ but something of that multiple.

b. $\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \times \begin{pmatrix} 4 \\ 2p \\ 1 \end{pmatrix} = k \begin{pmatrix} 4 \\ -p \\ 2 \end{pmatrix}$

use cross product given in formulae booklet.

$$\begin{array}{ccc} i & j & k \\ \hline 2 & 2 & -1 \\ 4 & 2p & 1 \end{array} = i\begin{vmatrix} 2 & -1 \\ 2p & 1 \end{vmatrix} - j\begin{vmatrix} 2 & -1 \\ 4 & 1 \end{vmatrix} + k\begin{vmatrix} 2 & 2 \\ 4 & 2p \end{vmatrix}$$

(Another way to find cross product is to find determinant 3x3 matrix)

$$= i\left[(2)(1) - (-1)(2p)\right] - j\left[(2)(1) - (-1)(4)\right] + k\left[(2)(2p) - (4)(2)\right]$$

$$= (2+2p)i - 6j + (4p-8)k$$

$$\begin{pmatrix} 2+2p \\ -6 \\ 4p-8 \end{pmatrix} = k\begin{pmatrix} 4 \\ -p \\ 2 \end{pmatrix}$$

$2+2p = 4k$ ①

$-6 = -kp$ ②

$4p-8 = 2k$ ③

Solve ① and ② simultaneously

$-6 = -kp$

$6 = kp$

$k = \frac{6}{p}$

$2+2p = 4k$

$2+2p = 4\left(\frac{6}{p}\right)$

$2+2p = \frac{24}{p}$

$2p + 2p^2 = 24$

$2p^2 + 2p - 24 = 0$

$p^2 + p - 12 = 0$

$(p+4)(p-3) = 0$

$p = 3$ or $p = -4$

Question 3 continued

c. $\Delta OAB = \frac{1}{2}\left|\underbrace{\overrightarrow{OA} \times \overrightarrow{OB}}\right| = 3\sqrt{2}$

use vector worked out in part (b)

$$\frac{1}{2}\left|\begin{pmatrix} 2+2p \\ -6 \\ 4p-8 \end{pmatrix}\right| = 3\sqrt{2}$$

$$\left|\begin{pmatrix} 2+2p \\ -6 \\ 4p-8 \end{pmatrix}\right| = 6\sqrt{2}$$

$\sqrt{(2+2p)^2 + (-6)^2 + (4p-8)^2} = 6\sqrt{2}$ square both sides.

$(2+2p)^2 + (-6)^2 + (4p-8)^2 = 72$

$4p^2 + 8p + 4 + 36 + 16p^2 - 64p + 64 = 72$

$20p^2 - 56p + 104 = 72$

$20p^2 - 56p + 32 = 0$

$5p^2 - 14p + 8 = 0$

$(5p-4)(p-2) = 0$

$p = \frac{4}{5}$ or $p = 2$

4. The velocity v ms^{-1}, of a raindrop, t seconds after it falls from a cloud, is modelled by the differential equation

$$\frac{dv}{dt} = -0.1v^2 + 10 \qquad t \geqslant 0$$

Initially the raindrop is at rest. @ $t_0 = 0,\ v_0 = 0$

(a) Use two iterations of the approximation formula $\left(\frac{dy}{dx}\right)_n \approx \frac{y_{n+1} - y_n}{h}$ to estimate the velocity of the raindrop 1 second after it falls from the cloud.

In terms of v and t.

(5)

Given that the initial acceleration of the raindrop is found to be smaller than is suggested by the current model,

(b) refine the model by changing the value of one constant.

(1)

a. There are 2 iterations.

Raindrop: 0 - 1 second.

$$\left(\frac{dv}{dt}\right)_n \approx \frac{v_{n+1} - v_n}{h}$$

$\frac{1-0}{2} = \frac{1}{2}$

$v_{n+1} = v_n + h\left(\frac{dv}{dt}\right)_n$ ← more useful form for calculating iterations

$h = \frac{1}{2}$

unknown so must work out first.

$v_1 = v_0 + h\left(\frac{dv}{dt}\right)_0$

$\left(\frac{dv}{dt}\right)_0 = -0.1(0)^2 + 10 = 10$

$v_1 = (0) + \frac{1}{2}(10)$

$v_1 = 5$

$\left(\frac{dv}{dt}\right)_1 = -0.1(5)^2 + 10 = 7.5$

$v_2 = v_1 + h\left(\frac{dv}{dt}\right)_1$

$v_2 = 5 + \frac{1}{2}(7.5)$

$v_2 = 8.75$ ≈ 8.75 ms^{-1}

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Question 4 continued

b. $\frac{dv}{dt} = -0.1v^2 + 10$

make this no. smaller

but above 0.

(Total for Question 4 is 6 marks)

5. The rectangular hyperbola H has equation $xy = 36$

(a) Use calculus to show that the equation of the tangent to H at the point $P\left(6t, \frac{6}{t}\right)$ is

$$yt^2 + x = 12t$$

(3)

The point $Q\left(12t, \frac{3}{t}\right)$ also lies on H.

(b) Find the equation of the tangent to H at the point Q.

(2)

The tangent at P and the tangent at Q meet at the point R.

(c) Show that as t varies the locus of R is also a rectangular hyperbola.

(4)

a. $xy = 36$ $\xrightarrow{\text{compare coefficients}}$ $xy = c^2$

$c^2 = 36$

$c = \pm 6$ unsure of c value so check parametric form.

Conics

	~~Ellipse~~	~~Parabola~~	Hyperbola	Rectangular Hyperbola
Standard Form	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	$y^2=4ax$	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$xy=c^2$
Parametric Form	$(a\cos\theta, b\sin\theta)$	$(at^2, 2at)$	$(a\sec\theta, b\tan\theta)$ $(\pm a\cosh\theta, b\sinh\theta)$	$\left(ct, \frac{c}{t}\right)$
Eccentricity	$e<1$ $b^2=a^2(1-e^2)$	$e=1$	$e>1$ $b^2=a^2(e^2-1)$	$e=\sqrt{2}$
Foci	$(\pm ae, 0)$	$(a, 0)$	$(\pm ae, 0)$	$(\pm\sqrt{2}c, \pm\sqrt{2}c)$
Directrices	$x=\pm\frac{a}{e}$	$x=-a$	$x=\pm\frac{a}{e}$	$x+y=\pm\sqrt{2}c$
Asymptotes	none	none	$\frac{x}{a}=\pm\frac{y}{b}$	$x=0, y=0$

parametric form: $\left(ct, \frac{c}{t}\right) \equiv \left(6t, \frac{6}{t}\right)$

$\therefore$ $c = +6$

$y = \frac{36}{x} = 36x^{-1}$

$\frac{dy}{dx} = -36x^{-2} = -\frac{36}{x^2}$

convert into parametric form $[x = 6t]$

$\frac{dy}{dx} = \frac{-36}{(6t)^2} = \frac{-\cancel{36}}{\cancel{36}t^2} = -\frac{1}{t^2}$

$M_{tangent} = -1/t^2$

$y - \frac{6}{t} = -\frac{1}{t^2}(x - 6t)$

$t^2y - 6t = -1(x - 6t)$

$t^2y - 6t = -x + 6t$

$yt^2 + x = 12t$ // (Shown)

Question 5 continued

b. $\frac{dy}{dx} = \frac{-36}{x^2}$

convert into parametric form
$x = 12t$

$\frac{dy}{dx} = \frac{-36}{(12t)^2} = \frac{-36}{144t^2} = -\frac{1}{4t^2}$

$m_{\text{tangent}} = -\frac{1}{4t^2}$

$y - \frac{3}{t} = -\frac{1}{4t^2}(x - 12t)$ use $y - y_1 = m(x - x_1)$ form where (x_1, y_1) known to find eqⁿ of tangent.

$4t^2y - 12t = -(x - 12t)$

$4t^2y - 12t = -x + 12t$

$4t^2y + x = 24t$ //

c. equate tangent P and Q and solve for coordinates R.

$4yt^2 + 4x = 48t$
$4yt^2 + x = 24t$ ⊖

$3x = 24t$
$x = 8t$

$4yt^2 + (8t) = 24t$
$4yt^2 = 16t$
$yt^2 = 4t$
$yt = 4$
$y = \frac{4}{t}$

$(8t, \frac{4}{t})$ ⊲ point R

let $X = 8t$
$Y = \frac{4}{t}$ } now rearrange for t and equate to create eqⁿ in terms of x and y.

Question 5 continued

$t = \frac{x}{8}$

$t = \frac{4}{y}$

$\frac{x}{8} = \frac{4}{y}$ cross multiply

$xy = 32$ follows $xy = c^2$ so locus of R rectangular hyperbola

$\vec{OP}$ $\vec{OQ}$ $\vec{OR}$

6. The points P, Q and R have position vectors $\begin{pmatrix}1\\-2\\4\end{pmatrix}$, $\begin{pmatrix}3\\1\\-5\end{pmatrix}$ and $\begin{pmatrix}2\\0\\3\end{pmatrix}$ respectively.

(a) Determine a vector equation of the plane that passes through the points P, Q and R, giving your answer in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$, where λ and μ are scalar parameters.

(2)

(b) Determine the coordinates of the point of intersection of the plane with the x-axis.

(4)

a. plane eq^n : $r = \vec{OP} + \lambda\vec{PQ} + \mu\vec{PR}$

$\vec{PR} = \vec{PO} + \vec{OR}$

$= \begin{pmatrix}-1\\2\\-4\end{pmatrix} + \begin{pmatrix}3\\1\\-5\end{pmatrix} = \begin{pmatrix}2\\3\\-9\end{pmatrix}$

$\vec{PR} = \begin{pmatrix}2\\3\\-9\end{pmatrix}$

$\vec{PQ} = \vec{PO} + \vec{OQ}$

$= \begin{pmatrix}-1\\2\\-4\end{pmatrix} + \begin{pmatrix}2\\0\\3\end{pmatrix} = \begin{pmatrix}1\\2\\-1\end{pmatrix}$

$\vec{PQ} = \begin{pmatrix}1\\2\\-1\end{pmatrix}$

$r = \begin{pmatrix}1\\-2\\4\end{pmatrix} + \lambda\begin{pmatrix}2\\3\\-9\end{pmatrix} + \mu\begin{pmatrix}1\\2\\-1\end{pmatrix}$

b. @ x-axis plane, $y = 0$ and $z = 0$

write plane eq^n in general form:

$r = \begin{pmatrix}1 + 2\lambda + \mu\\-2 + 3\lambda + 2\mu\\4 - 9\lambda - \mu\end{pmatrix}$ } set these to 0 and solve simultaneously

to find λ and μ - then sub in to find x-intercept.

Question 6 continued

$1 + 2\lambda + \mu = x$ ①

$-2 + 3\lambda + 2\mu = 0$ ②

$4 - 9\lambda - \mu = 0$ ③

Solving ② and ③ simultaneously,

③ × 2 : $8 - 18\lambda - 2\mu = 0$

$-2 + 3\lambda + 2\mu = 0$
$8 - 18\lambda - 2\mu = 0$ ⊕ (add both eqⁿs to get rid of μ term)

$6 - 15\lambda = 0$

$15\lambda = 6$

$\lambda = 6/15 = 2/5$

$\lambda = 2/5$ ← sub back into ③ to solve for μ.

$4 - 9\left(\frac{2}{5}\right) - \mu = 0$

$\frac{2}{5} - \mu = 0$

$\mu = 2/5$

Now sub λ and μ values back into gn eqⁿ.

$$\begin{pmatrix} 1 + 2\left(\frac{2}{5}\right) + \left(\frac{2}{5}\right) \\ -2 + 3\left(\frac{2}{5}\right) + 2\left(\frac{2}{5}\right) \\ 4 - 9\left(\frac{2}{5}\right) - \left(\frac{2}{5}\right) \end{pmatrix} = \begin{pmatrix} 11/5 \\ 0 \\ 0 \end{pmatrix}$$

$\left(\frac{11}{5}, 0, 0\right)$

7.

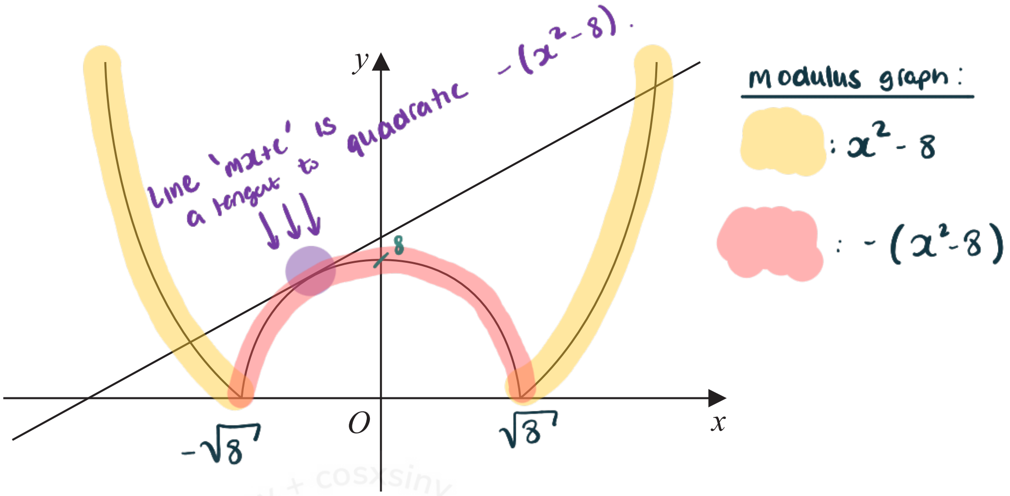

Figure 1

Figure 1 shows a sketch of the curve with equation $y = |x^2 - 8|$ and a sketch of the straight line with equation $y = mx + c$, where m and c are positive constants.

The equation

$$|x^2 - 8| = mx + c$$

$x^2 - 8 = 0$
$x^2 = 8$
$x = \pm\sqrt{8}$ ← roots

has exactly 3 roots, as shown in Figure 1.

(a) Show that

$$m^2 - 4c + 32 = 0$$

(2)

Given that $c = 3m$

(b) determine the value of m and the value of c

only 1 value for each

(3)

(c) Hence solve

$$|x^2 - 8| \geqslant mx + c$$

(3)

a. $-(x^2-8) = mx+c$

$-x^2+8 = mx+c$

$x^2 + mx + c-8 = 0$

@ tangent, only 1 intersection so $b^2-4ac=0$

$(1)x^2 + (m)x + (c-8) = 0$

a b c

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Question 7 continued

$(m)^2 - (4)(1)(c-8) = 0$

$m^2 - 4c + 32 = 0$ // (Shown)

b. If $c = 3m$

$m^2 - 4(3m) + 32 = 0$

$m^2 - 12m + 32 = 0$

$(m-4)(m-8) = 0$

$m = 4$ or $m = 8$

$\downarrow \qquad \downarrow$

$c = 3(4) = 12 \qquad c = 3(8) = 24$

2 lines to pick : $y = 4x + 12$

$y = 8x + 24$

Sub into $|x^2 - 8| = mx + c$

$\downarrow$

into negated $\cap$ curve section

This is to find intersection point.

(1) $8 - x^2 = 4x + 12$

$x^2 + 4x + 4 = 0$

$(x+2)^2 = 0$

$x = -2$

$y = 4(-2) + 12$

$y = 4$

$(-2, 4)$

(2) $8 - x^2 = 8x + 24$

$x^2 + 8x + 16 = 0$

$(x+4)^2 = 0$

$x = -4$

$y = 8(-4) + 24$

$y = -8$

$(-4, -8)$

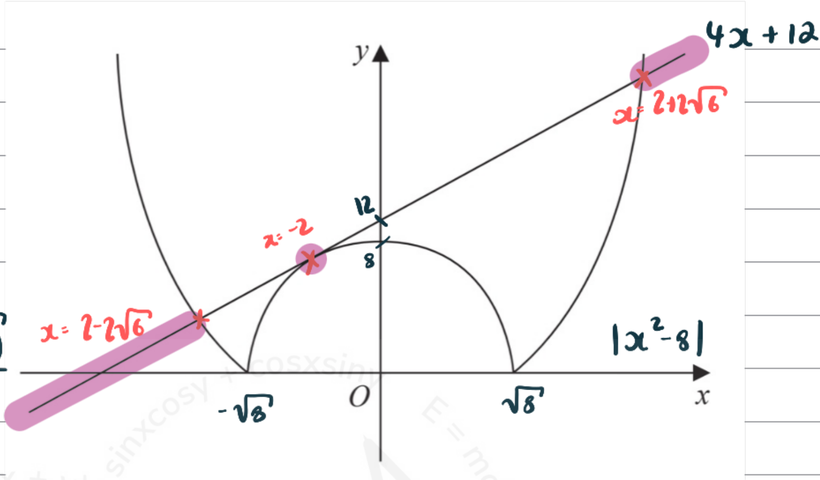
$\uparrow$

not possible as this is below x-axis but intersection is above.

Question 7 continued

$\therefore\ m = 4,\ c = 12$

c. $|x^2 - 8| \geq 4x + 12$



$x^2 - 8 = 4x + 12$

$x^2 - 4x - 20 = 0$

$x = \dfrac{-(-4) \pm \sqrt{(-4)^2 - (4)(1)(-20)}}{2(1)}$

$x = 2 \pm 2\sqrt{6}$

$8 - x^2 = 4x + 12$

$x^2\ 4x + 4 = 0$

$(x+2)^2 = 0$

$x = -2$

$x \leq 2 - 2\sqrt{6},\ x = -2,\ x \geq 2 + 2\sqrt{6}$ //

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8. The Taylor series expansion of $f(x)$ about $x = a$ is given by

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots + \frac{(x-a)^r}{r!}f^{(r)}(a) + \ldots$$

(i) (a) Use differentiation to determine the Taylor series expansion of $\ln x$, in ascending powers of $(x-1)$, up to and including the term in $(x-1)^2$

(4)

(b) Hence prove that

$$\lim_{x \to 1}\left(\frac{\ln x}{x-1}\right) = 1$$

(2)

(ii) Use L'Hospital's rule to determine

$$\lim_{x \to 0}\left(\frac{1}{(x+3)\tan(6x)\operatorname{cosec}(2x)}\right)$$

(*Solutions relying entirely on calculator technology are not acceptable.*)

(4)

a. $f(x) = \ln(x)$ $\quad f(1) = \ln(1) = 0$

$f'(x) = \frac{1}{x} = x^{-1}$ $\quad f'(1) = \frac{1}{1} = 1$

$f''(x) = -x^{-2}$ $\quad f''(1) = -(1)^{-2} = -1$

$f(x) = 0 + (x-1)(1) + \frac{(x-1)^2}{2!}(-1) + \ldots$

$f(x) = (x-1) - \frac{1}{2}(x-1)^2 + \ldots$

b. $\lim_{x \to 1}\left(\frac{(x-1) - \frac{1}{2}(x-1)^2 + \ldots}{(x-1)}\right)$

$= \lim_{x \to 1}\left(1 - \frac{1}{2}(x-1) + \ldots\right)$

as $x \to 1$, $\frac{1}{2}(x-1) \to 0$, every subsequent term $\to 0$.

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Question 8 continued

$\therefore \lim_{x\to 1}\left(\frac{\ln(x)}{x-1}\right) = 1$

ii. rewrite as: $\lim_{x\to 0}\left(\frac{\sin(2x)}{(x+3)\tan(6x)}\right)$

can only use L'Hopital's rule if $\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right)$ lead to $\frac{0}{0}$ or $\pm\infty$

$\sin(2(0)) = 0$

$((0)+3)\tan(6(0)) = 0$

$\therefore \frac{0}{0}$ ← can use L'Hopital's

$f(x) = \sin(2x)$ $\quad g(x) = (x+3)\tan(6x)$

$f'(x) = 2\cos(2x)$ $\quad g'(x) = \tan(6x) + (x+3)(6\sec^2(6x))$

$\lim_{x\to 0}\left(\frac{2\cos(2x)}{\tan(6x)+(x+3)(6\sec^2(6x))}\right)$

$= \frac{2\cos(2(0))}{\tan(6(0)) + ((0)+3)(6\sec^2(6(0)))}$ sub in 0 for x and evaluate

$= \frac{2}{0+(3)(6)} = \frac{2}{18}$

$= 1/9$ //

9. A particle P moves along a straight line.

At time t minutes, the displacement, x metres, of P from a fixed point O on the line is modelled by the differential equation

$$t^2\frac{d^2x}{dt^2} - 2t\frac{dx}{dt} + 2x + 16t^2x = 4t^3\sin 2t \qquad \text{(I)}$$

(a) Show that the transformation $x = ty$ transforms equation (I) into the equation

$$\frac{d^2y}{dt^2} + 16y = 4\sin 2t$$

(5)

(b) Hence find a general solution for the displacement of P from O at time t minutes.

(8)

a. $x = ty$

use product rule

differentiate both sides w.r.t. t

$\frac{dx}{dt} = y + t\frac{dy}{dt}$

use product rule

differentiate both sides w.r.t. t

$\frac{d^2x}{dt^2} = \frac{dy}{dt} + \frac{dy}{dt} + t\frac{d^2y}{dt^2}$

sub in $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$ to create new eq^n in terms of y, $\frac{dy}{dt}$, $\frac{d^2y}{dt^2}$.

$t^2\left(2\frac{dy}{dt} + t\frac{d^2y}{dt^2}\right) - 2t\left(y + t\frac{dy}{dt}\right) + 2ty + 16t^2(ty) = 4t^3\sin(2t)$

$\cancel{2t^2\frac{dy}{dt}} + t^3\frac{d^2y}{dt^2} - \cancel{2ty} - \cancel{2t^2\frac{dy}{dt}} + \cancel{2ty} + 16t^3y = 4t^3\sin(2t)$

$\cancel{t^3}\frac{d^2y}{dt^2} + 16\cancel{t^3}y = 4\cancel{t^3}\sin(2t)$ (divide both sides by t^3)

$\frac{d^2y}{dt^2} + 16y = 4\sin(2t)$ //

Question 9 continued

b. Solve auxillary eq^n.

$m^2 + 16 = 0$

$m^2 = -16$

$m = \pm 4i$

$y = A\cos(4t) + B\sin(4t)$ (complementary function C.F.)

Let particular integral (P.I): $y = \lambda \sin(2t)$

$y' = 2\lambda \cos(2t)$

$y'' = -4\lambda \sin(2t)$

$(-4\lambda \cancel{\sin(2t)}) + 16(\lambda \cancel{\sin(2t)}) = 4\cancel{\sin(2t)}$ (sub P.I. into 2nd order D.E and solve for λ)

$-4\lambda + 16\lambda = 4$

$12\lambda = 4$

$\lambda = 1/3$

P.I: $y = \frac{1}{3}\sin(2t)$

general sol^n: C.F. + P.I.

$y = A\cos(4t) + B\sin(4t) + \frac{1}{3}\sin(2t)$

$x = ty$

$y = \frac{x}{t}$

must convert back in terms of x and t as x is displacement.

$\frac{x}{t} = A\cos(4t) + B\sin(4t) + \frac{1}{3}\sin(2t)$

$x = t\left[A\cos(4t) + B\sin(4t) + \frac{1}{3}\sin(2t)\right]$